

EXAMPLE: PROBLEM 2.1.28 OF THE TEXT

110.302 DIFFERENTIAL EQUATIONS  
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**Problem.** Consider the Initial Value Problem

$$y' + \frac{2}{3}y = 1 - \frac{1}{2}t, \quad y(0) = y_0.$$

Find the value of  $y_0$  for which the solution touches, but does not cross, the  $t$ -axis.

**Strategy.** This ODE is linear, so the general solution can be found by using an integrating factor. We solve for the general solution and then write the general solution in terms of the initial value  $y_0$ . Then we use the properties of the particular solution we are looking for to locate the proper value of  $y_0$ .

**Solution.** This ODE is linear and already in the standard form to read off the coefficient function of  $y$  in the equation. We get  $p(t) = \frac{2}{3}$ . Thus the integration factor is

$$e^{\int p(t) dt} = e^{\int \frac{2}{3} dt} = e^{\frac{2}{3}t}.$$

Multiply the ODE through by this integrating factor and

$$\begin{aligned} e^{\frac{2}{3}t} \left[ y' + \frac{2}{3}y = 1 - \frac{1}{2}t \right] \\ e^{\frac{2}{3}t}y' + \frac{2}{3}e^{\frac{2}{3}t}y = e^{\frac{2}{3}t} - \frac{1}{2}te^{\frac{2}{3}t} \\ \frac{d}{dt} \left[ e^{\frac{2}{3}t}y \right] = e^{\frac{2}{3}t} - \frac{1}{2}te^{\frac{2}{3}t}. \end{aligned}$$

Now integrate with respect to  $t$  to get

$$\begin{aligned} \int \frac{d}{dt} \left[ e^{\frac{2}{3}t}y \right] dt &= \int \left( e^{\frac{2}{3}t} - \frac{1}{2}te^{\frac{2}{3}t} \right) dt \\ e^{\frac{2}{3}t}y &= \frac{3}{2}e^{\frac{2}{3}t} - \frac{1}{2} \left( \frac{3}{2}te^{\frac{2}{3}t} - \frac{9}{4}e^{\frac{2}{3}t} \right) + C. \end{aligned}$$

Solving for  $y(t)$ , we get

$$y(t) = \frac{3}{2} - \frac{1}{2} \left( \frac{3}{2}t - \frac{9}{4} \right) + Ce^{-\frac{2}{3}t} = \frac{21}{8} - \frac{3}{4}t + Ce^{-\frac{2}{3}t}.$$

This is a form of the general solution to the ODE, a 1-parameter family of solutions parameterized by the constant of integration  $C$ . However, we can also parameterize this family of curves directly by the initial data  $y(0) = y_0$ . This is better, as it ties the curves directly to their  $y$ -intercept here. Here

$$y(0) = y_0 = \frac{21}{8} - \frac{3}{4}(0) + Ce^{-\frac{2}{3}(0)} = \frac{21}{8} + C,$$

so that we can write our general solution directly using the initial data as

$$y(t) = \frac{21}{8} - \frac{3}{4}t + \left(y_0 - \frac{21}{8}\right) e^{-\frac{2}{3}t}.$$

At right are some representative curves for different values of  $y_0$ . Notice that it does look like there will be a curve that will touch, but not cross, the  $t$ -axis. We are looking for the value of  $y_0$  of this particular curve.

Anytime one is trying to pick out a particular curve in a parameterized family of them, one must make use of distinguishing data carefully. Here, we are not given points on the curve directly, which is usually the case. Instead, we are given certain properties. Firstly, the curve touches the  $t$ -axis at some point. We do not know where, so we can write  $y(t_0) = 0$ , where  $t_0$  is the intersection of the curve with the  $t$ -axis, and the  $y$ -coordinate there would be zero. Secondly, since the curve must be smooth, and touches but does not cross the  $t$ -axis, the curve will have to be tangent to the curve at  $t_0$ . So the second piece of data is  $y'(t_0) = 0$ . So we have

$$(1) \quad y(t_0) = 0 = \frac{21}{8} - \frac{3}{4}t_0 + \left(y_0 - \frac{21}{8}\right) e^{-\frac{2}{3}t_0},$$

$$(2) \quad y'(t_0) = 0 = -\frac{3}{4} - \frac{2}{3} \left(y_0 - \frac{21}{8}\right) e^{-\frac{2}{3}t_0}.$$

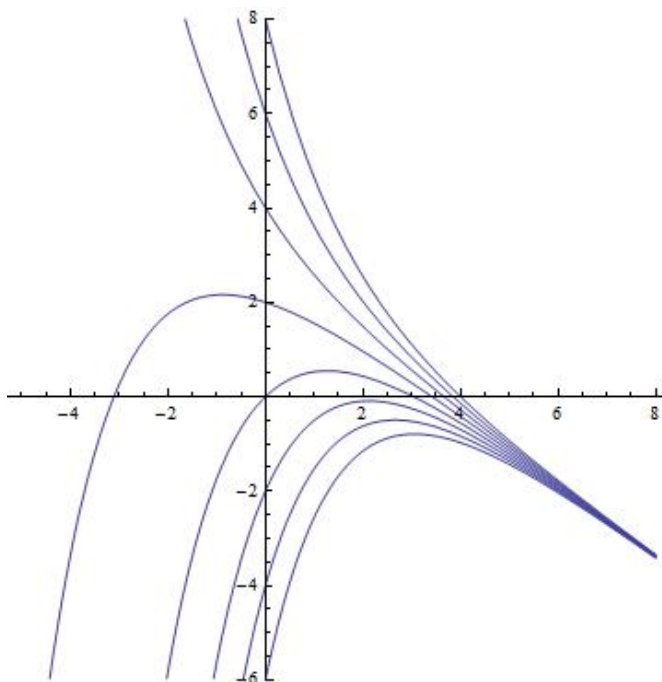


FIGURE 1. A few representative curves  $y(t)$ .

This is now two (non-linear) equations in two unknowns, which we can solve to find the particular value of  $y_0$ .

One way to do this: Rearrange Equation 2 to get

$$(3) \quad -\frac{9}{8} = \left(y_0 - \frac{21}{8}\right) e^{-\frac{2}{3}t_0}.$$

Notice that in Equation 1 of the pair, a good piece of the RHS matches the RHS here. Substitute in the LHS of this last equation into Equation 1 to get

$$\begin{aligned} y(t_0) = 0 &= \frac{21}{8} - \frac{3}{4}t_0 + \left(y_0 - \frac{21}{8}\right) e^{-\frac{2}{3}t_0} \\ 0 &= \frac{21}{8} - \frac{3}{4}t_0 + \left(-\frac{9}{8}\right) \\ &= \frac{12}{8} - \frac{3}{4}t_0 = 0. \end{aligned}$$

This leaves  $t_0 = 2$  as a solution.

To find the corresponding  $y_0$ , substitute this value of  $t_0$  back into either Equation 1 or 2. In Equation 2, we get

$$\begin{aligned} 0 &= -\frac{3}{4} - \frac{2}{3} \left(y_0 - \frac{21}{8}\right) e^{-\frac{2}{3}(2)} \\ -\frac{9}{8}e^{\frac{4}{3}} + \frac{21}{8} &= y_0 \cong -1.64288. \end{aligned}$$

Below, we have added the graph of the curve corresponding to this value of  $y_0$ . As you can see, it fits the criteria and solves the problem.

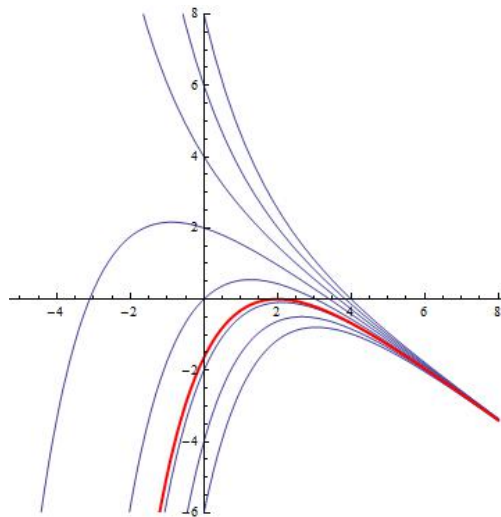


FIGURE 2. The solution curve  $y(t)$  is in red.